Modes of Convergence

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- We recall some definitions regarding convergence

<u>Def:</u> (X, μ) measure space.

i) (f_n) ~~seq~~ sequence of measurable functions on X is an <u>L^p-Cauchy sequence</u> if $\forall \varepsilon > 0$, $\exists N$ st $\forall i,j > N$, $\|f_i - f_j\|_{L^p} = \left(\int_X |f_i - f_j|^p d\mu\right)^{1/p} < \varepsilon$.

ii) (f_n) as above <u>converges in L^p</u> to a measurable function f if $\lim_n \|f_n - f\|_{L^p} = \lim_n \left(\int_X |f_n - f|^p\right)^{1/p} = 0$

iii) $f_n \to f$ <u>pointwise μ-almost everywhere</u> if $\exists$ set $N \in X$ st $\mu(N) = 0$ and $\forall x \in N^c$, $f_n(x) \to f(x)$.

iv) $f_n \to f$ pointwise; $\forall x \in X, \forall \varepsilon > 0, \exists N$ st $\forall n \geq N$, $|f_n(x) - f(x)| < \varepsilon$.

$\triangle$

<u>Recall:</u> • pointwise $\not\Rightarrow$ L^1 convergence

Take $f_n = n\, 1_{[0, 1/n]}$. Then $f_n \to 0$ p.w. but

$$\lim_n \|f_n - 0\|_{L^1} = \lim_n \int_0^{1/n} n \, d\mu = 1 \neq 0.$$

• $L^1 \not\Rightarrow$ pointwise (a.e.)

Take $(E_n) = \underset{E_1}{[0,1]}, \underset{E_2}{[0, 1/2]}, \underset{E_3}{[1/2, 1]}, \underset{E_4}{[0, 1/3]}, [1/3, 2/3], [2/3, 1], \dots$

Then (1_{E_n}) measurable on $[0,1]$ ~~st~~

$\longrightarrow$

$\Rightarrow \lim_n \| \mathbb{1}_{E_n} - 0 \|_{L^1} = \lim_n \mu(E_n) = 0$

$\therefore \mathbb{1}_{E_n} \to 0$ in L^1. But $\forall x \in [0,1]$, $(\mathbb{1}_{E_n}(x)) \not\to 0$

as each x is in infinitely many E_n. ◊

Note: Convergence in L^1 is a.e. unique, that is if $f_n \to f$ in L^1 and $f_n \to g$ in L^1, then $f = g$ a.e.

$\hookrightarrow \|f-g\|_{L^1} \leq \underbrace{\|f_n - f\|_{L^1}}_{\to 0} + \underbrace{\|f_n - g\|_{L^1}}_{\to 0} \quad \forall n$

Prop: $(f_n) \to f$ in $L^1 \Rightarrow \exists$ subsequence of f_n that converges (pointwise) a.e.

Idea: L^1 conv $\Rightarrow L^1$ Cauchy $\Rightarrow$ Subseq. (f_{n_k}) of bd L^1 variation $\Rightarrow$ ~~1b~~ $(f_{n_k}) \to g$ pointwise a.e. and in L^1

As $f_{n_k} \to f$ in $L^1 \Rightarrow f = g$ a.e $\Rightarrow f_{n_k} \to f$ p.a.e. □

! Lemma: If $\sum_{n=1}^{\infty} \int_X |f_{n+1} - f_n| \, d\mu < \infty \Big\}$ a.k.a. bd. L^1 variation.

Then $(f_n) \to f$ in L^1 and p.a.e.

Idea: $g = \sum_{n=1}^{\infty} |f_{n+1} - f_n| \in L^1$ by MCT $\Rightarrow (f_n(x))$ bd var. $\forall x$

$\Rightarrow$ p.a.e. conv. to some f

Then use DCT w/ $|f - f_n| \leq g$. ◊

Prop: (L^1-DCT) (f_n) sequence of L^1 fktns $\to f$ pointwise. ← measurable
If $\exists\ g \in L^1(X)$ st $|f_n| \leq g \ \forall n \in \mathbb{N}$ then
$$f_n \to f \text{ in } L^1$$

Pf As $|f_n| \leq g \ \forall n$ and $f_n \to f$ point. it gives $|f| \leq g$, thus $|f_n - f| \leq |f_n| + |f| = 2g \ \forall n$.
By DCT,
$$\lim_n \int_X |f_n - f| \, d\mu = \int \lim_n |f_n - f| \, d\mu = 0$$

Thus $f_n \to f$ in L^1 $\square$

Note: This is stronger than the claim of the DCT, that is $f_n \to f$ in L^1 implies
$$\lim_n \int_X f_n \, d\mu = \int_X f \, d\mu$$

↳ $f_n \to f$ in $L^1 \Rightarrow f \in L^1$ and
$$\left| \int_X f_n \, d\mu - \int_X f \, d\mu \right| \leq \int_X |f_n - f| \, d\mu = \underbrace{\|f_n - f\|_{L^1}}_{\xrightarrow[n]{} 0}$$

Note: For a finite measure space ($\mu(X) < \infty$) and $1 \leq p \lneq q \leq \infty$
By Hölder's inequality we have
$$L^q(X) \subsetneq L^p(X)$$

Hölder: $\frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \|f \cdot g\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}$

Note: With $p = q = 2$, $\|f g\|_{L^1} \leq \|f\|_{L^2} \cdot \|g\|_{L^2}$

Recall (from lecture)

Ⓣ 3.6: "Best approximation" to f (wrt to L^2-norm) is given by the Fourier series.

Ⓣ 3.12: The Fourier series of a periodically continued L^2 function is pointwise convergent almost everywhere.

! Ⓣ 3.6 $\not\Rightarrow$ Ⓣ 3.12 !

Convergence in $L^2([0,1]) \subset L^1([0,1])$ does not imply p.a.e convergence

In general, L^1-convergence does not give anything better than a pointwise a.e. convergent subsequence

Ex: With $\hat{f}(t) := \int_{\mathbb{R}} e^{-itx} f(x)\, dx$ show, for $f \in L^2(\mathbb{R})$ and $c > 0$, that

$$\lim_{c \to \infty} \int_{-c}^{c} f(x) e^{-ixt}\, dx$$

exists in the L^2 sense and is precisely $\hat{f}(t)$.

$\Longrightarrow$

We define $f_{c_n}(x) := \begin{cases} f(x) & |x| \le c_n \\ 0 & \text{else} \end{cases}$

for any $(c_n) \to \infty$. Then $f_{c_n} \to f$ pointwise

and is dominated by $f \in L^2$ by assumption.

Then by DCT, $f_{c_n} \xrightarrow{L^2} f$,

thus Cauchy in L^2.

Next we use the following classical theorem

(T) (Plancherel's) Let $f \in L^1 \cap L^2$

The Fourier transform is an isometry, ie

$$\|\hat{f}\|_{L^2} = \|f\|_{L^2}$$

Sketch: By inverse FT $f(x) = \int_{\mathbb{R}} \hat{f}(t) e^{ixt} dt$

$$\Rightarrow \int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} \overline{f(x)} \int_{\mathbb{R}} \hat{f}(t) e^{ixt} dt\, dx$$

$$\overset{\text{Fubini}}{=} \int_{\mathbb{R}} \hat{f}(t) \int_{\mathbb{R}} \overline{f(x)} e^{ixt} dx\, dt$$

$$= \int_{\mathbb{R}} |\hat{f}(t)|^2 dt$$

Then extends to $f \in L^2$ by density of $L^1 \cap L^2$ $\Diamond$

$\longrightarrow$

$\therefore$ By Plancherel

$$\|\widehat{f_{c_n} - f_{c_m}}\|_{L^2} = \|\hat{f}_{c_n} - \hat{f}_{c_m}\|_{L^2} = \|f_{c_n} - f_{c_m}\|_{L^2}$$

the sequence $(\hat{f}_{c_n})$ is Cauchy in L^2 thus converges in L^2.

Thus $\lim_n \hat{f}_{c_n} = \lim_{c \to \infty} \hat{f}_c = \hat{f}$ by construction.

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" Fourier transform converges in the L^2 norm "

Q: What is the purpose / motivation for the Fourier transform?

A: It ~~does~~ converts a function into a form that describes the frequencies of the original function. For example, decomposing a sound into the intensities of its pitches.

In particular, it converts your function / signals into a different domain; in this new domain, difficult problems may be more easily solved.