

Modes of Convergence

27.11.2023

We recall some definitions regarding convergence

Def: (X, μ) measure space.

i) (f_n) ~~seq~~ sequence of measurable fctns on X

is an L^p -Cauchy sequence if $\forall \varepsilon > 0$,

$$\exists N \text{ st } \forall i, j > N, \|f_i - f_j\|_{L^p} = \left(\int_X |f_i - f_j|^p d\mu \right)^{1/p} < \varepsilon$$

ii) (f_n) as above converges in L^p to a measurable

$$\text{fctn } f \text{ if } \lim_n \|f_n - f\|_{L^p} = \lim_n \left(\int_X |f_n - f|^p \right)^{1/p} = 0$$

iii) $f_n \rightarrow f$ ^{pointwise} μ -almost everywhere if $\exists$ set $N \in \mathcal{X}$

st $\mu(N) = 0$ and $\forall x \in N^c, f_n(x) \rightarrow f(x)$.

iv) $f_n \rightarrow f$ pointwise; $\forall x \in X, \forall \varepsilon > 0, \exists N \text{ st } \forall n \geq N, |f_n(x) - f(x)| < \varepsilon$

Recall: • pointwise $\nrightarrow L^2$ convergence

Take $f_n = n \cdot 1_{[0, 1/n]}$. Then $f_n \rightarrow 0$ p.w. but

$$\lim_n \|f_n - 0\|_{L^2} = \lim_n \sqrt{\int_0^{1/n} n^2 d\mu} = 1 \neq 0.$$

! • $L^2 \nrightarrow$ pointwise (a.e.)

Take $(E_n) = [0, 1], [0, 1/2], [1/2, 1], [0, 1/3], [1/3, 2/3], [2/3, 1], \dots$

Then (1_{E_n}) measurable on $[0, 1]$ ~~that~~

~~lim~~

$\rightarrow$

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$$\Rightarrow \lim_n \|I_{E_n} - 0\|_{L^2} = \lim_n \mu(E_n) = 0$$

$\therefore I_{E_n} \rightarrow 0$ in L^2 . But $\forall x \in [0,1]$, $(I_{E_n}(x)) \not\rightarrow 0$
as each x is in infinitely many E_n . $\square$

Note: Convergence in L^2 is a.e. unique, that is
if $f_n \rightarrow f$ in L^2 and $f_n \rightarrow g$ in L^2 ,
then $f=g$ a.e.

$$\hookrightarrow \|f-g\|_{L^2} \leq \underbrace{\|f_n - f\|_{L^2}}_{\rightarrow 0} + \underbrace{\|f_n - g\|_{L^2}}_{\rightarrow 0} \quad \forall n$$

Prop: $(f_n) \rightarrow f$ in $L^2 \Rightarrow \exists$ subsequence of f_n that
converges (pointwise) a.e.

Idea: L^2 conv $\Rightarrow L^2$ Cauchy $\Rightarrow$ Subseq. (f_{n_k}) of bd L^2
variation $\Rightarrow \exists (f_{n_k}) \rightarrow g$ pointwise a.e. and in L^2

As $f_{n_k} \rightarrow f$ in $L^2 \Rightarrow f=g$ a.e. $\Rightarrow f_{n_k} \rightarrow f$ p.a.e. $\square$

Lemma: If $\left\{ \sum_{n=1}^{\infty} \int_X |f_{n+1} - f_n| d\mu < \infty \right\}$ a.k.a. bd L^2 variation.

Then $(f_n) \rightarrow f$ in L^2 and p.a.e.

Idea: $g = \sum_{n=1}^{\infty} |f_{n+1} - f_n| \in L^1$ by MCT $\Rightarrow [f_n(x)]$ bd var. $\forall x$

$\Rightarrow$ p.a.e. conv. to some f

Then use DCT w/ $|f - f_n| \leq g$. $\square$

Prop: (L^2 -DCT) (f_n) sequence of L^2 fctns $\rightarrow f$ ^{measurable} pointwise.

If $\exists g \in L^2(X)$ st $|f_n| \leq g \quad \forall n \in \mathbb{N}$ then

$$f_n \rightarrow f \text{ in } L^2$$

Pf: As $|f_n| \leq g \quad \forall n$ and $f_n \rightarrow f$ point. it gives $|f| \leq g$,
thus $|f_n - f| \leq |f_n| + |f| = 2g \quad \forall n$.

By DCT,

$$\lim_n \int_X |f_n - f| d\mu = \int \lim_n |f_n - f| d\mu = 0$$

Thus $f_n \rightarrow f$ in L^2 □

Note: This is stronger than the claim of the DCT,
that is $f_n \rightarrow f$ in L^2 implies

$$\lim_n \int_X f_n d\mu = \int_X f d\mu$$

$\hookrightarrow f_n \rightarrow f$ in $L^2 \Rightarrow f \in L^2$ and

$$\left| \int_X f_n d\mu - \int_X f d\mu \right| \leq \int_X |f_n - f| d\mu = \|f_n - f\|_{L^1} \xrightarrow{n \rightarrow \infty} 0$$

Note: For a finite measure space ($\mu(X) < \infty$) and $1 \leq p \leq q \leq \infty$

By Hölder's inequality we have

$$L^q(X) \subset L^p(X)$$

$$\text{Hölder: } \frac{1}{p} + \frac{1}{q} = 1 \Rightarrow \|f \cdot g\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}$$

Note: with $p=q=2$, $\|f \cdot g\|_{L^1} \leq \|f\|_{L^2} \cdot \|g\|_{L^2}$

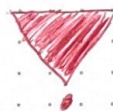
Recall (from lecture)

③ 3.6: "Best approximation" to f (wrt to L^2 -norm)
is given by the Fourier series.

③ 3.12: The Fourier series of a periodically
continued L^2 function is pointwise convergent
almost everywhere.



③ 3.6 $\rightarrow$ ③ 3.12



Convergence in $L^2([0,1]) \subset L^1([0,1])$

does not imply p.a.e convergence

In general, L^2 -convergence does not give anything
better than a pointwise convergent subsequence $\triangleleft$
a.e.

~~Ex:~~ With $\hat{f}(t) := \int_{\mathbb{R}} e^{-itx} f(x) dx$ show, for

$f \in L^2(\mathbb{R})$ and $c > 0$, that

$$\lim_{c \rightarrow \infty} \int_{-c}^c f(x) e^{-ixt} dx$$

exists in the L^2 sense and is precisely $\hat{f}(t)$.

$\Rightarrow$

We define $f_{c_n}(x) := \begin{cases} f(x) & |x| \leq c_n \\ 0 & \text{else} \end{cases}$

for $\forall c_n \rightarrow \infty$. Then $f_{c_n} \rightarrow f$ pointwise

and B dominated by $f \in L^2$ by assumption.

Then by DCT, $f_{c_n} \xrightarrow{L^2} f$,
thus Cauchy in L^2 .

Next we use the following classical theorem

① (Plancherel's) Let $f \in L^1 \cap L^2$

The Fourier transform B on $\mathbb{R}$ ometry, i.e.

$$\|\hat{f}\|_{L^2} = \|f\|_{L^2}$$

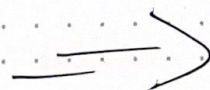
Sketch: By inverse FT $f(x) = \int_{\mathbb{R}} \hat{f}(t) e^{ixt} dt$

$$\Rightarrow \int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(t) e^{ixt} dt dx$$

$$\stackrel{\text{Fubini}}{=} \int_{\mathbb{R}} \hat{f}(t) \int_{\mathbb{R}} f(x) e^{ixt} dx dt$$

$$= \int_{\mathbb{R}} |\hat{f}(t)|^2 dt$$

Then extends to $f \in L^2$ by density of $L^1 \cap L^2$ $\diamond$



$\therefore$ By Plancherel

$$\|f_{c_n} - f_{c_m}\|_{L^2} = \|\hat{f}_{c_n} - \hat{f}_{c_m}\|_{L^2} = \|f_{c_n} - f_{c_m}\|_{L^2}$$

the sequence $(\hat{f}_{c_n})$ is Cauchy in L^2 thus
converges in L^2 .

Thus $\lim_n \hat{f}_{c_n} = \lim_{c \rightarrow \infty} \hat{f}_c = \hat{f}$ by construction.

"Fourier transform converges in the L^2 norm"

Q: What is the purpose/motivation for the
Fourier transform?

A: It ~~like~~ converts a function into a form
that describes the frequencies of the original
function. For example, decomposing a sound
into the intensities of its pitches.

In particular, it converts your function/signals
into a different domain; in this new domain
difficult problems may be more easily solved.

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